

The Bright Side of Mathematics

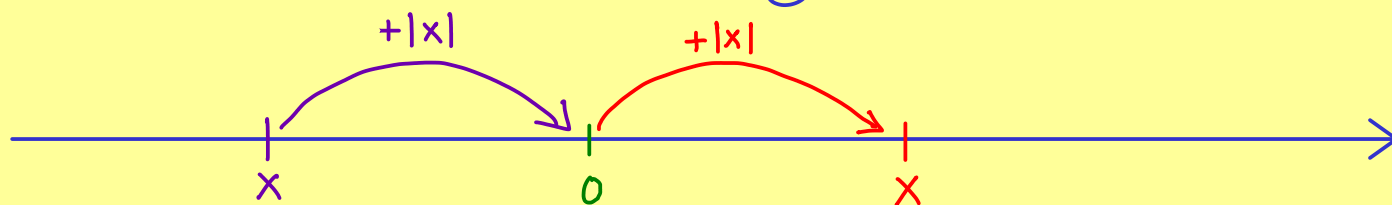


Start Learning Reals - Part 1

Real numbers $\mathbb{R}$

Starting point: $\mathbb{Q}$ is the set of fractions $\leadsto$ field and Archimedean order $\leq$
 $x > 0$, $x < 0$

Absolute value: For $x \in \mathbb{Q}$ define: $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



How far away is x from 0 ? $\leadsto |x|$

Problem: There is no $x \in \mathbb{Q}$ with $x^2 = 2$

$$x_1 = \frac{14}{10} = \frac{7}{5} \leadsto x_1^2 = \frac{49}{25} \approx 2$$

$$x_2 = \frac{141}{100} \leadsto x_2^2 = \frac{19881}{10000} \approx 2$$

$$x_3 = \frac{1414}{1000} \leadsto x_3^2 = \frac{499849}{250000} \approx 2$$

$$x_4 = \frac{14142}{10000} \leadsto x_4^2 = \frac{49999041}{25000000} \approx 2$$

$$x_5 = \frac{141421}{100000} \leadsto x_5^2 = \frac{19999899241}{10000000000} \approx 2$$

$\vdots$

$\vdots$

$\vdots$

$x = ?$

$\leadsto x^2 = 2$

distance:

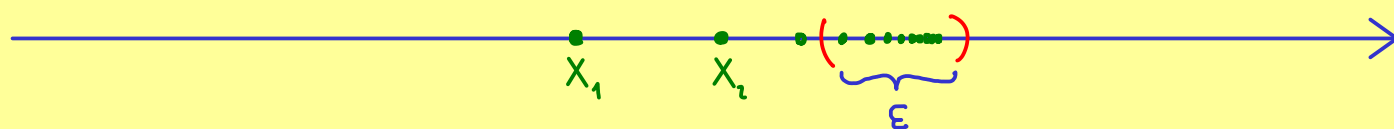
$$|x_5 - x_2|$$

We consider a sequence $(x_n)_{n \in \mathbb{N}}$ (infinite list; formally: a map $\mathbb{N} \rightarrow \mathbb{Q}$, $n \mapsto x_n$)

with the property:

$$\forall \varepsilon \in \mathbb{Q} \quad \exists N \in \mathbb{N} \quad \forall n, m \in \mathbb{N} : \left(\varepsilon > 0 \wedge n, m \geq N \implies |x_n - x_m| < \varepsilon \right)$$

In short: $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m \geq N : |x_n - x_m| < \varepsilon \quad (*)$



Cauchy sequence: sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{Q}$ and property $(*)$