



The Bright Side of Mathematics

Banach fixed-point theorem

Let (X, d) be a complete metric space

metric = distance function

all Cauchy sequences are convergent

and $\Phi: X \rightarrow X$ be a contraction, which means:

$$\exists q \in [0, 1) \quad \forall x, \tilde{x} \in X: d(\Phi(x), \Phi(\tilde{x})) \leq q \cdot d(x, \tilde{x})$$

Then: Φ has a unique fixed point $x^* \in X$ ($\Phi(x^*) = x^*$)

and for each $x_0 \in X$ we have: $\Phi^n(x_0) \xrightarrow{n \rightarrow \infty} x^*$.

Proof: For a given $x_0 \in X$, define $x_n := \Phi^n(x_0)$

Is $(x_n)_{n \in \mathbb{N}}$ a Cauchy sequence?

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\Phi(x_n), \Phi(x_{n-1})) \leq q \cdot d(x_n, x_{n-1}) \\ &= q \cdot d(\Phi(x_{n-1}), \Phi(x_{n-2})) \leq q^2 \cdot d(x_{n-1}, x_{n-2}) \\ &\dots \leq q^n \cdot d(x_1, x_0) \quad (\text{proof by induction}) \end{aligned}$$



$$\text{For } n > m: d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + d(x_{m+1}, x_m)$$

Δ -inequality

$$\leq (q^{n-1} + q^{n-2} + \dots + q^m) \cdot d(x_1, x_0)$$

$$= q^m \cdot \sum_{k=0}^{n-1-m} q^k \cdot d(x_1, x_0)$$

$$\leq \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

$$\leq \frac{q^m}{1-q} \cdot d(x_1, x_0)$$

$$\Rightarrow (x_n)_{n \in \mathbb{N}} \text{ is a Cauchy sequence} \quad (d(x_n, x_m) \xrightarrow{n, m \rightarrow \infty} 0)$$

completeness

$$\Rightarrow (x_n)_{n \in \mathbb{N}} \text{ has a unique limit } x^* \in X$$

$$\text{Fixed point?} \quad \Phi(x^*) = \Phi(\lim_{n \rightarrow \infty} x_n) \stackrel{\text{contraction is continuous}}{=} \lim_{n \rightarrow \infty} \Phi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$$

Uniqueness? We have a map Φ with $d(\Phi(x), \Phi(\tilde{x})) \leq q \cdot d(x, \tilde{x})$ and fixed points $x^*, \hat{x}$.

$$x^* \neq \hat{x} \Rightarrow d(x^*, \hat{x}) = d(\Phi(x^*), \Phi(\hat{x})) \leq q \cdot d(x^*, \hat{x})$$

$$\Rightarrow 1 \leq q$$

$$\text{By contraposition: } 0 \leq q < 1 \Rightarrow x^* = \hat{x} \quad \square$$